Write your name and student number in the top left corner of each page

1. Determine the *z*-transform of the following sequences and their respective ROCs including the zeros and poles (if applicable):

   (a) $x[n]=\left(-\frac{1}{3}\right)^n \mu[n]-\left(\frac{1}{2}\right)^n \mu[-n-1]$

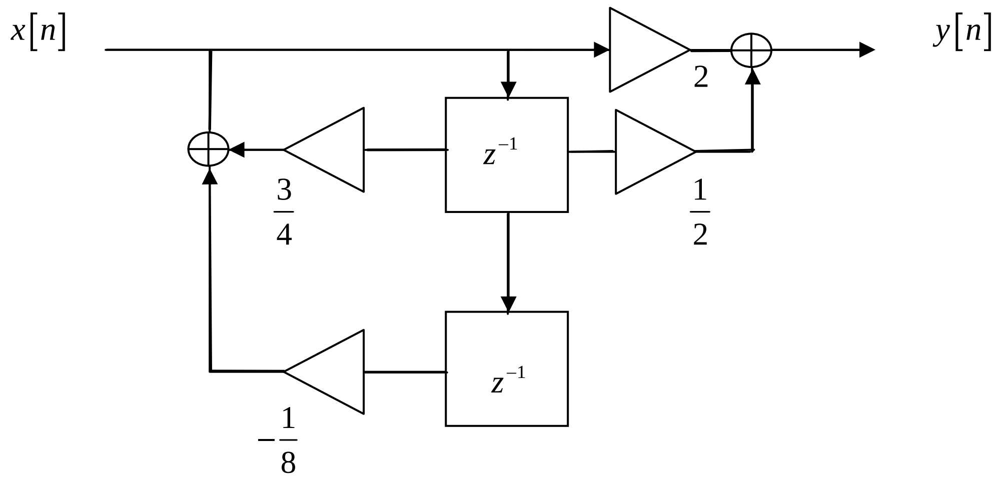
   (b) $x[n]=\left(-\frac{1}{2}\right)^n \mu[n-1]$

2. Determine the Impulse Response $h[n]$ which satisfies the following linear constant-coefficient difference equation:

   $y[n]-\frac{1}{2}y[n-1]=x[n]-\frac{1}{4}x[n-1]$

   (hint: set $x[n]=\delta[n]$ and determine $h[n]$ via Fourier Transformations)

3. Consider the following LTI system:




   (a) write down the transfer function of the system $H(z)=\frac{Y(z)}{X(z)}$

   (b) determine the ROC, poles and zeros and plot their positions in a zero-pole plot

4. Determine the DTFT of the following sequences:

   (a) $x[n]=\alpha^n\left(\mu[n]-\mu[n-8]\right)$ for $|\alpha|<1$

   (b) $x[n]=n\alpha^n\mu[n]$ for $|\alpha|<1$

5. Determine the 4-point circular convolution of the two length-4 sequences $g[n]=\{1,2,0,1\}$ and $h[n]=\{2,2,1,1\}$. Draw these functions and the convolution result.

Solutions for the exam of 25.01.2010:

1.

a) $x[n] = \left(-\frac{1}{3}\right)^n \mu[n] - \left(\frac{1}{2}\right)^n \mu[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left[\left(-\frac{1}{3}\right)^n \mu[n]z^{-n} - \left(\frac{1}{2}\right) \mu[-n-1]z^{-n}\right] =$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \frac{1}{z^n} - \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n \frac{1}{z^n} = |n = -m| =$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n z^{-n} - \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{-m} z^m = \sum_{n=0}^{\infty} \left(-\frac{1}{3}z^{-1}\right)^n - \sum_{m=1}^{\infty} \left(\frac{1}{2}^{-1} z\right)^m =$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{3}z^{-1}\right)^n + 1 - \sum_{m=0}^{\infty} \left(\frac{1}{2}^{-1} z\right)^m =$$

$$= \frac{1}{1 - \left(-\frac{1}{3}\right) z^{-1}} + 1 - \frac{1}{1 - \left(\frac{1}{2}\right)^{-1} z} \qquad (*)$$

From (*) we determine the ROC as:

$|\left(-\frac{1}{3}\right) z^{-1}| < 1$ which gives us: $|z| > \frac{1}{3}$ and

$|\left(\frac{1}{2}\right)^1 z| < 1$ which gives us: $|z| < \frac{1}{2}$

The intersection of these two regions in the complex plane is the ROC. The unit circle is outside the ROC, so this sequence does not have a DTFT. The fact that the ROC is inside the unit circle tells us that this is an acausal sequence; the second term in the initial expression confirms that.

Now, to determine the poles and zeros, we rewrite (*) as:

$$\frac{1}{1 - \left(-\frac{1}{3}\right) z^{-1}} + 1 - \frac{1}{1 - \left(\frac{1}{2}\right)^{-1} z} =$$

$$= \frac{3z}{3z+1} + 1 - \frac{1}{1-2z} = \frac{3z(1-2z) + (3z+1)(1-2z) - 3z - 1}{(3z+1)(1-2z)} =$$

$$= \frac{z(1-12z)}{(3z+1)(1-2z)}$$

We can see that there are two poles at $z = -\frac{1}{3}$ and $z = \frac{1}{2}$ as well as two zeros at: $z = 0$ and $z = \frac{1}{12}$.

b)

$$x[n] = \left(-\frac{1}{2}\right)^n \mu[n-1]$$

$$X(z) = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} =$$

$$= -1 + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} = -1 + \frac{1}{1 - \left(-\frac{1}{2}\right) z^{-1}}$$

The ROC is found as: $|-\frac{1}{2z}| < 1$, so the ROC is: $|z| > \frac{1}{2}$
This is a causal sequence; the unit circle is inside the ROC.
To find the poles and zeros, we rewrite as:

$$-1 + \frac{1}{1 - \left(-\frac{1}{2}\right) z^{-1}} = \frac{-1}{2z+1}$$

there is one pole located at $z = -\frac{1}{2}$ and there are no zeros.

2.

$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$$

$$Y(z) - Y(z)\frac{1}{2}z^{-1} = X(z) - X(z)\frac{1}{4}z^{-1} =$$

$$H(z) = \frac{Y(z)}{X(x)} = \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}} =$$

$$= \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{1}{4}\frac{1}{1 - \frac{1}{2}z^{-1}}z^{-1}$$

$$h[n] = \left(\frac{1}{2}\right)^n \mu[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} \mu[n-1] = \left(\frac{1}{2}\right)^n \mu[n] - \left(\frac{1}{2}\right)^{n+1} \mu[n-1]$$

Alternatively, we can substitute $x[n] = \delta[n]$, then take the DTFT of both sides, divide the right side with the left and do the IDTFT to recover $h[n]$.

3.

a)

$$H(z) = \frac{2 + \frac{1}{2}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{4z(4z+1)}{8z^2 - 6z + 1}$$

b)

To solve for the poles, we find the solutions of the quadratic in the denominator. They are: $z = \frac{1}{2}, z = \frac{1}{4}$. The zeros are: $z = 0, z = -\frac{1}{4}$.

A property of the ROC is that it is outside of the outermost pole (for a causal system). So, the ROC is: $|z| > \frac{1}{4}$. The unit circle is in the ROC (the DTFT exists), and the poles are inside the unit circle so the system is stable. From the pole - zero diagram we can see that this is a Low-pass filter.

4.

a)

$$x[n] = \alpha^n(\mu[n] - \mu[n-8])$$

$$X(\omega) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} - \sum_{n=8}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1 - \alpha e^{-j\omega}} - \frac{(\alpha e^{-j\omega})^8}{1 - \alpha e^{-j\omega}}$$

b)

$$x[n] = n\alpha^n \mu[n]$$

Use the property of the Z transform: $Z(ng(n)) = -z\frac{dG(z)}{dz}$ where $g(n) = \alpha^n$ and since we are on the unit circle, $z = e^{j\omega}$.

$$Z(n\alpha^n) = -\frac{d}{de^{j\omega}}\left[\frac{1}{1 - \alpha e^{-j\omega}}\right]e^{j\omega} = \frac{\alpha e^{j\omega}}{(e^{j\omega} - \alpha)^2}$$

5.

$$g[n] = \{1, 2, 0, 1\}$$

$$h[n] = \{2, 2, 1, 1\}$$

Circular shift and multiply to obtain the convolution result. We will shift $g[n]$:

$$g[n] * h[n] = \frac{1102}{2211}, \frac{2110}{2211}, \frac{0211}{2211}, \frac{1021}{2211} = \{6, 7, 6, 5\}$$

Alternatively, we could have taken the DFT of both sequences, multiplied the results and then do the IDFT to obtain the convolution. Or, use the circular convolution matrices.